Absolute and convective instabilities of the natural convection in a vertical heated slot

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The spatiotemporal instability of a natural convection flow in vertical heated slot is studied theoretically. The two unstable modes, secondary cell and traveling wave, are illustrated to be absolute and convective instabilities, respectively. Using a model to simulate the temperature gradient in the center of the slot, we propose an interpretation of the mechanism controlling the reverse transition of flow patterns, and explain the temperature fluctuation observed after the reverse transition in terms of the traveling wave mode.

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Natural convection in a closed cavity provides a rich system for the study of pattern-forming instabilities and transition phenomena from laminar flow to turbulence. One of the best-studied examples is the flow in vertical slot heated sidewalls, the convection in a vertical fluid layer with a horizontal temperature gradient. When the slot is tall enough and the Prandtl number is small, the flow in the center portion of the slot is almost parallel and the horizontal temperature gradient is nearly constant, so heat is transferred by conduction from hot to cold wall. Batchelor [1] first analyzed the flow in a slot with infinite aspect ratio and named it the conduction regime. The instability of the conduction regime was studied theoretically and experimentally [2,3] and a temporal mode was used in all instability analyses. The main interests of both experiments and theoretical researches were centered on the critical values, and two neutral curves or two unstable modes were found in the first stage of transition to turbulence: secondary cell and traveling wave. However, one of the simplest theoretical questions has not yet been addressed clearly: whether the temporal mode is a proper choice in the instability analysis for both unstable modes.

Generally, a velocity profile is called absolute unstable if localized disturbances can spread both upstream and downstream and contaminate the entire flow eventually; in contrast, the profile will be called convective unstable if the disturbances are swept away from the source. Therefore, if we study the instability characteristics of an unstable mode not only in the neutral condition but also in the amplifying state, it becomes necessary to determine the properties of the unstable mode, absolute instability or convective instability, then the temporal mode or spatial mode would be used in the corresponding analysis. In closed flows, such as Rayleigh-Bernard convection or Taylor-Couette flow, the critical Reynolds numbers of convective and absolute instability coincide because of the reflection symmetry [4]. For the above reasons, it becomes attractive to study the instability characteristics of the unstable modes in the convection of vertical heated slot.

Recently, emphasis has been placed on the convection flow in the vertical isothermal heated slot with a finite aspect ratio because of its geophysical and technological importance [5–11]. Numerical researches have provided enough results of such convection flow in laminar manner, especially on the flow patterns of the secondary cells near the core of the slot. The flow remains to be unicellular when the Grashof number is smaller than a critical value. As the Grashof number increases larger than the value the convection flow becomes substantially unsteady and then the unicellular convection breaks down into a multicellular convection arranged in a series of secondary cells. When the Grashof number further increases, the number of cells decreases in the center region and the flow becomes unsteady again and returns to a unicellular structure. This is called reverse transition, and has been observed in experimental researches [12,13]. In addition, when the aspect ratio is too small, only the unicellular pattern can be observed, so in fact there is a critical aspect ratio for a given Prandtl number. To summarize, though we have obtained comparatively abundant data about the flow pattern during the reverse transition, the insight into the controlling mechanism is still quite rudimentary. The present paper represents an attempt to determine the absolute and convective properties of the unstable modes and interpret the phenomena during the reverse transition.

The convection flow considered here is located in a narrow vertical slot with two isothermal vertical walls at different temperatures $+\theta$ and $-\theta$. A Cartesian coordinate system is fixed at the midplane of the slot in such a way that the positive Z direction is vertical, opposite in direction to the gravity g. The sidewalls are at $x=\pm h$, where $2h$ is the width of the slot. If the Boussinesq and parallel approximations are made, the governing equations can be resolved in terms of the velocity U, temperature T_0, coordinate distance X, and time τ, nondimensionalized by $g\gamma\theta h^2/\nu$, θ, h, and $\nu/gh\gamma\theta$, respectively. For the infinite aspect ratio case, the base flow takes the form as below, which was first presented by Batchelor:

$$U=1/6(X^3-X), \quad T_0=-X, \tag{1}$$

where γ and ν is the volumetric coefficient of thermal expansion and kinematic viscosity. The stream function and temperature of disturbance components are taken to be in the form

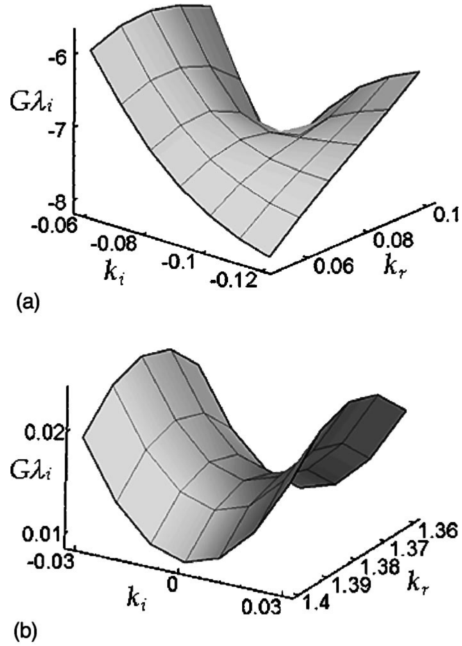

FIG. 1. The λ_i surface according to the k plane for infinite aspect ratio Pr=12.5. (a) The traveling wave mode for $G=2000$. (b) The secondary cell mode for $G=494$, which is larger slightly than the critical value. The λ_i surface is symmetrical about $k_i=0$, and at the saddle point $k_i=0$, $k_r=1.38$.

$$\psi(X,Z,\tau)=\varphi(X)e^{-i\lambda\tau+ikZ}, \tag{2}$$

$$T(X,Z,\tau)=\Theta(X)e^{-i\lambda\tau+ikZ}. \tag{3}$$

Here λ and k are both taken as complexes. The Orr-Sommerfield equation coupled with energy equation can be obtained according to stability theory:

$$\Delta\Delta\varphi+ikG(H\varphi+c\Delta\varphi)+\Theta'=0, \tag{4}$$

$$Pr^{-1}\Delta\Theta+ikG(T_0'\varphi-U\Theta+c\Theta)=0. \tag{5}$$

Boundary conditions

$$\Theta=\varphi=\varphi'=0, \quad X=\pm 1, \tag{6}$$

where $\Delta=\partial^2/\partial X^2-k^2$, $H\varphi=U''\varphi-U\Delta\varphi$, and Grashof number $G=g\gamma\theta h^3/\nu^2$, $c=\lambda/k$, $\mathrm{Pr}=\nu/a$, the Rayleigh number is defined as Ra=16G Pr, a is the molecular thermal diffusivity. The six-order eigensystem defined by Eqs. (4)–(6) is resolved by a fourth-order finite-difference scheme. In the present work, the criterion of decision is

$$|\lambda_{n+1}-\lambda_n|<10^{-8}.$$

A step size of $\Delta X=0.04$ was used and checked by recomputing several points on the neutral curve with $\Delta X=0.02$, the new value agreed with the old one to better than 0.1 %. As a result, the step size of $\Delta X=0.04$ was used in all calculations.

According to the stability theory, the distinction between absolute and convective instabilities can be made by studying the long-time (τ) evolution of the response of the flow at a fixed spatial location to an impulsive excitation applied at the origin. Based on the method of steepest descent, for $\tau\to\infty$ and fixed X, the response is ultimately dominated by the mode with zero group velocity [4]

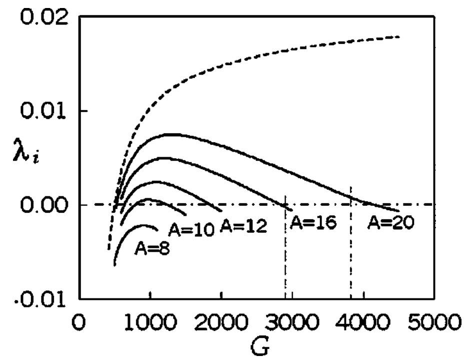

FIG. 2. The largest amplifying rate λ_i of the secondary cell mode as a function of the Grashof number for Pr=0.71 in the core of a vertical slot with different aspect ratios (solid lines) and with infinite aspect ratio (thick dash line). The vertical dash lines indicate the positions of the unicellular flow pattern observed in numerical studies [10,11].

$$\left.\frac{d\lambda}{dk}\right|_{k=k_0}=0, \tag{7}$$

where λ and k are both complexes, the subscript 0 denotes the saddle point of the complex function $\lambda(k)$. $\lambda_0=\lambda(k_0)$ is commonly called the absolute frequency. According to the Briggs criterion: if the corresponding $\lambda_{0,i}>0$, the flow is absolutely unstable; otherwise, it is convectively unstable.

The saddle points in the complex k plane for Pr=12.5 at $G=2000$ and 494 are shown in Fig. 1. The λ_i at the saddle points are -0.0036 and 2.785×10^{-5}, respectively. It is shown clearly that the secondary cell mode is absolute instability even at the initial unstable stage. Therefore, the temporal mode is suitable for its stability analysis (k is taken as real). Whereas the traveling wave mode is convective instability even at large Grashof number, the spatial mode should be a more satisfactory choice than temporal mode. Apparently, unlike the Rayleigh-Benard convection, the absolute and convective instabilities of such closed flow have their own critical Grashof numbers.

Eckert and Calson [2] studied experimentally the temperature distribution in air with an interferometer at different aspect ratio A (height of the slot L/width $2h$). They found that when the Grashof number increased to a critical value (G_a), a vertical temperature gradient β (nondimensionalized by θ and h) developed in the core of the vertical slot, and then maintained at a constant value after the Grashof number exceeded another one (G_b). This phenomenon has been confirmed by later research [14,15]. In Elder's experiments [14], it was found that after a nearly logarithmic increase, the value of βA in the core became independent of Grashof number and just a weak function of Prandtl number. In his studies, $\beta A\to0.5$ for paraffin, 0.55 for silicone, and the results of Eckert and Calson for air give 0.6. To our knowledge, most of the detailed numerical research was centered on the slot of air with aspect ratio 20 [6,11] and 16 [9,10]. It should be meaningful to study the influence of the temperature gradient on the flow patterns during the reverse transi-

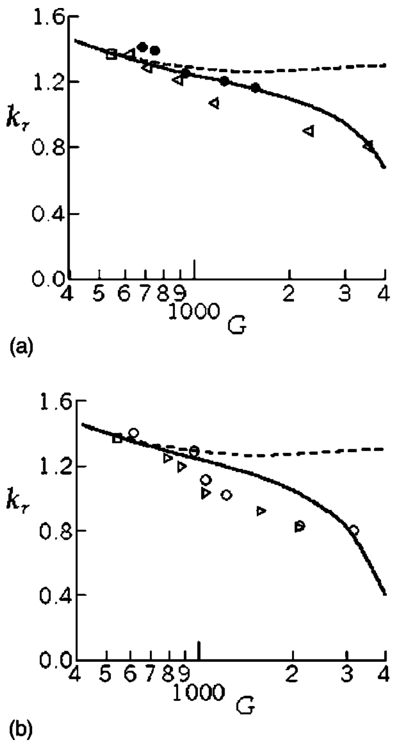

FIG. 3. The wave number of the secondary cell mode with the largest amplifying rate as a function of the Grashof number for (a) $A=20$ and (b) $A=16$, $\mathrm{Pr}=0.71$. The solid and the dash lines correspond to the flow in the core of a slot with finite aspect ratio and infinite aspect ratio respectively. □ denotes the critical value observed experimentally by Vest and Arpaci [3],●, ○, ▷, ◁ denote the results of numerical simulations [6,9–11], respectively.

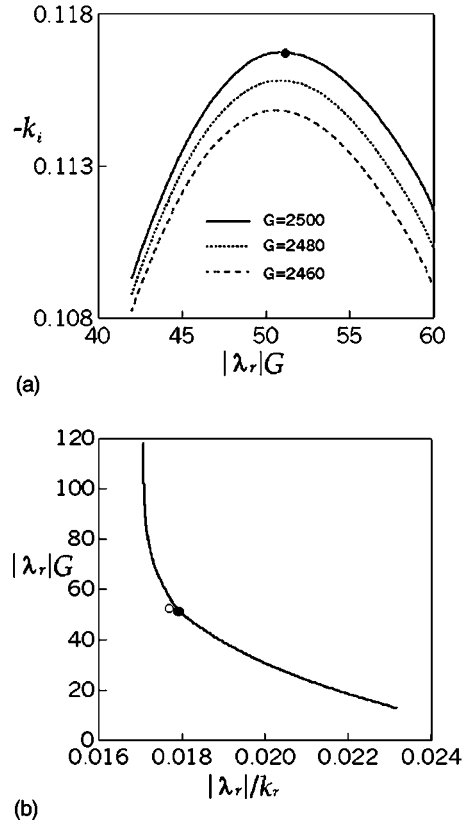

FIG. 4. (a) The amplifying rate of the traveling wave mode as a function of the nondimensional frequency at different Grashof numbers; (b) characteristics of the traveling wave mode at $G=2500$. ● and ○ indicate the positions of the traveling wave mode with the largest amplifying rate and the temperature fluctuation found in the numerical simulations of Lee and Korpela, respectively. Both numerical and theoretical analyses aimed at a slot of water with an aspect ratio $A=25$, $\mathrm{Pr}=6.7$.

tion, so based on the experimental results [2], a model was built for air to simulate the temperature gradient at the core of the slot:

$$\beta A=\begin{cases} 0, G<G_a, \\ 0.6[\ln(G)-\ln(G_a)]/[\ln(G_b)-\ln(G_a)], \\ G_a<G<G_b, \\ 0.6, G>G_b, \end{cases} \tag{8}$$

where $\ln(G_a)=4.456+0.376\ln(A)$ and $\ln(G_b)=7.15+0.376\ln(A)$. When the Grashof number is larger than G_a the base flow in the center of the vertical slot takes the following forms which were first suggested by Elder [14], where U and T_0 are the real parts of

$$U=-\frac{i}{M^2}[f_1(X)-f_{-1}(X)],$$

$$T_0=-\frac{1}{2}[f_1(X)+f_{-1}(X)], \tag{9}$$

where

$$f_n(X)=\frac{\sinh[(1+ni)M(1+X)/2]-\sinh[(1+ni)M(1-X)/2]}{\sinh[(1+ni)M]},$$

$$n=\pm 1.$$

The temperature stratification parameter $M=(\beta\mathrm{Ra}/4)^{1/4}$. Since the secondary cell unstable mode is absolute instability, temporal mode is used in the following corresponding analysis.

As shown in Fig. 2, the largest amplifying rate of the secondary cell unstable mode for the infinite aspect ratio case has a monotone increase along with the Grashof number, so the multicellular structure would always exist as long as the Grashof number is larger than the critical value. Whereas in the core of the slot with finite aspect ratio the largest amplifying rate reaches a maximum then decreases to be less than zero, this result predicts definitely that there is a reverse transition of flow patterns from multicellular structure to a unicellular one. The critical Grashof numbers corresponding to the end of the reverse transition agree well with numerical results. It can also be concluded from Fig. 2 that the smaller the aspect ratio A is, the earlier the reverse transition happens and the smaller the maximum is. When the aspect ratio $A=10$, unstable mode almost disappears. This value is smaller slightly than the critical aspect ratios obtained from numerical simulations 11–12 [5], 10–12.5 [6], and 11.5 [11].

The primary character of the reverse transition is the decrease of the wave number or the increase of the wavelength.

This trend coincides with the results of instability analysis (Fig. 3). On the contrary, for the infinite aspect ratio case the wave number of the secondary cell increases slightly after an initial decrease. Except for the numerical data from Vest and Arpaci, others were calculated by measuring the wavelengths of cells near the core of the slot. It can also be found that the wave numbers show a comparatively larger variation in a range of the Grashof number 1000–2500 for both $A=16$ and 20. This is connected with the fact that in this range the secondary cell solutions are time-periodic asymptotic ones, and depend heavily on the initial conditions (starting from a motionless and isothermal state or gradually increasing/decreasing the Grashof number) [6,10]. A better agreement can be expected if the model used to simulate the revolution of the temperature gradient β is closer to the conditions in the numerical studies. According to the above discussions, we can get a basically comprehensive understanding of the mechanism controlling the reverse transition.

Lee and Korpela [6] calculated the transient solution of the secondary cell during the reverse transition of an isothermal vertical heated slot of water with an aspect ratio of 25 at $G=2500$. According to their numerical results, the cells moved upwards in the upper half of the slot and downwards in the lower half, then a unicellular structure was left in the core of the slot. Shortly after that the transient was over and a quasiperiodic solution for temperature field was left. Corresponding solutions have also been found in slot of air [11]. The linear analysis of Bergholz was tried to interpret this phenomenon qualitatively [16]. But one must bear in mind the fact that the base flow solution investigated by Bergholz is characterized by a linear increase of the wall temperature, so the boundary conditions are different from those of numerical researches. More importantly, in his theoretical analysis the stratification parameter M was looked as constant when the critical parameters were resolved. This treatment means that the temperature gradient β would decrease as the Grashof number increases. Apparently, this contradicts the results of experiments.

According to the calculations of Lee and Korpela, the nondimensional wave speed and frequency λ_r of the temperature fluctuation in the center of the slot are 0.0176 and 0.02092, respectively, the corresponding nondimensional maximum vertical velocity is 0.0184. Based on the maximum vertical velocity and the model presented above, it can be determined that the temperature gradient β is 0.473. As with the infinite aspect ratio case, it is easy to ascertain that the traveling wave unstable mode in a slot with finite aspect ratio is also convective instability, so spatial mode is used in the following analysis (λ is treated as real).

It is shown clearly in Fig. 4 that at $G=2500$ the nondimensional frequency of the temperature fluctuation observed in numerical simulations is very close to the value of traveling wave mode with the largest amplifying rate. According to the present model, the nondimensional wave speed of the traveling wave mode with the largest amplifying rate is 0.0179, which agrees very well with the numerical simulation value 0.0176, so it can be concluded confidently that the temperature fluctuation found after the reverse transition belongs to the traveling wave mode.

More importantly, according to our instability analysis, there are a couple of traveling wave solutions at the center of the slot, which have the same values of wavelength and wave speed but move oppositely in the vertical direction. Consequently it is reasonable to consider that the cells' up and down movements observed in Lee and Korpela's numerical simulations are connected with the traveling wave solutions. Usually, the convective instability mode is not considered so important as the absolute instability one in closed flows. However, as discussed above, in the natural convection of vertical isothermal heated slot, the absolute and convective instabilities may exist simultaneously, and especially, the convective unstable mode can dominate the flow patterns after the absolute unstable mode-secondary cell disappears. These characteristics make such convection flow distinct from other closed flow systems.

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